Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Miracles *versus* Nature: being an application of certain propositions in the theory of chances to the Christian miracles. By Protimalethes. Cambridge, 1847, 8vo.

The theory, as may be supposed, is carried further than most students of the subject would hold defensible.

An astronomical Lecture. By the Rev. R. Wilson. Greenock, 1847, 12mo.

Against the moon's rotation on her axis.

[Handed about in the streets in 1847: I quote the whole:] Important discovery in astronomy, communicated to the Astronomer Royal, December 21st, 1846. That the Sun revolve round the Planets in $25748\frac{2}{5}$ years, in consequence of the combined attraction of the planets and their satellites, and that the Earth revolve round the Moon in 18 years and 228 days. D. T. Glazier [altered with a pen into Glazion.] Price one penny.

1847. In the *United Service Magazine* for September, 1847, Mrs. Borron, of Shrewsbury, published some remarks tending to impeach the fact that Neptune, the planet found by Galle, really was the planet which Leverrier and Adams had a right to claim. This was followed (September 14) by two pages, separately circulated, of *Further Observations upon the Planets Neptune and Uranus, with a Theory of Perturbations;* and (October 19, 1848) by three pages of *A Review of M. Leverrier's Exposition.* Several persons, when the remarkable discovery was made, contended that the planet actually discovered was an intruder; and the future histories of the discovery must contain some account of this little after-piece. Tim Linkinwater's theory that there is no place like London for coincidences, would have been utterly overthrown in favour of what they used to call the celestial spaces, if there had been a planet which by chance was put near the place assigned to Neptune at the time when the discovery was made.

(*To be continued.*)

On the Value of Annuities payable Half-yearly, Quarterly, &c.

By T. B. Sprague, Esq., M.A., *Actuary of the Equity and Law Life Assurance Society.*

THIS subject has been so fully treated by many writers, that it may be thought little or nothing new can be said in reference to it. It may, however, be useful to bring together and to compare the results obtained by the standard writers on the theory of life contingencies, to examine how far those results agree with each other, and when they differ, to ascertain the causes of the difference.

When an annuity of £1 is payable m times in the course of the year, the common rule for finding its value is to add $\frac{m-1}{2m}$ to the value of the yearly annuity. This may be easily arrived at in the following way. The age of the life on which the annuity depends being k, the value of an annuity of £1, of which the first payment is made at once, is $1+a_k$; and the value of the same annuity when the first payment is made at the end of a year, is a_k; therefore the value of the same annuity when the first payment is made at the end of the $\frac{1}{m}$th part of a year, will be $a_k+\frac{m-1}{m}$ approximately. If there are m such annuities of which the first payments are made at the end of the $\frac{1}{m}$th, $\frac{2}{m}$th, $\frac{3}{m}$th . . . . portions of a year respectively, the sum of their values will be approximately

$$\left(a_k+\frac{m-1}{m}\right)+\left(a_k+\frac{m-2}{m}\right)+\left(a_k+\frac{m-3}{m}\right)+\ .\ .\ .\ +a_k$$
$$=ma_k+\frac{(m-1)m}{2m}.$$

Whence, dividing by m, the value of an annuity of £1 payable by instalments of $\frac{1}{m}$ at equal intervals throughout each year, is $a_k+\frac{m-1}{2m}$ approximately.

This demonstration does not enable us to judge of the closeness of the approximation; but this is equally true of Mr. Milne's demonstration, which is very much more lengthy without being more satisfactory. Mr. Milne begins by finding the value of an annuity according to De Moivre's hypothesis—first, when payable annually; secondly, when payable by m equal instalments; and proves that the latter value is greater then the former by $\frac{m-1}{2m}$ approximately. He then establishes this "Lemma":—"If q and q' be any two quantities, known or unknown, that are nearly equal, and p, p', their respective near values, both determined by the same method of approximation, then will $p-p'$ be much nearer to $q-q'$ than either of the proximate values is to the true." From this it follows that the true value of the annuity when payable m times a year exceeds the true value of the annuity payable yearly by $\frac{m-1}{2m}$ nearly.

In reference to this rule, Professor De Morgan remarks (*Companion to the Almanac* for 1842, p. 9):—“When an annuity is to be paid, not yearly, but half-yearly or quarterly, &c., *interest being convertible into principal only once a year*, the common correction of A, the present value of a yearly annuity, is to write $A+\frac{n-1}{2n}$, instead of A, n being the number of times which the annuity is payable yearly. This correction is strictly true when we speak of a perpetual annuity, on the supposition that by interest being convertible once a year, we mean that money may be invested at any one time of the year, so that the grantor of the annuity must be held to lose simple interest for the rest of the year on each of the prepaid portions.” The limitation expressed by the words which I have placed in italics, is, however, unnecessary; for the correction applies equally whether the interest be convertible only once a year or n times a year.

Mr. Baily has not considered the question generally, but has given the exact formulæ for the values of annuities payable half-yearly and quarterly. His formula for the value of an annuity payable half-yearly, which is based upon the usual supposition that the deaths are distributed uniformly over each year, is

$$\frac{\sqrt{v}}{4}\{2(1+i)^{\frac{1}{2}}+2+i\}a_k+\frac{\sqrt{v}}{4},$$

which is equal to

$$\frac{a_k}{4}\left\{2+\sqrt{1+i}+\frac{1}{\sqrt{1+i}}\right\}+\frac{1}{4\sqrt{1+i}}$$

$$=\frac{a_k}{4}\left\{2+1+\frac{i}{2}-\frac{i^2}{8}+\frac{i^3}{16}-\ \ldots\ldots\right.$$

$$\left.+1-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\ \ldots\ldots\right\}$$

$$+\frac{1}{4}\left\{1-\frac{i}{2}+\frac{3}{8}i^2-\frac{5}{16}i^3+\ \ldots\ldots\right\}$$

$$=\frac{a_k}{4}\left\{4+\frac{i^2}{4}-\frac{i^3}{4}+\ \ldots\ldots\right\}+\frac{1}{4}-\frac{i}{8}+\frac{3}{32}i^{\cancel{3}\,2}-\ \ldots\ldots$$

$$=a_k+\frac{1}{4},\ \text{approximately},$$

since i is usually a small fraction.

The above expansions enable us to estimate the closeness of the approximation given by the formula $a_k + \frac{1}{4}$. The errors caused by taking the first terms only of the series

$$4 + \frac{i^2}{4} - \frac{i^3}{4} + \dots$$

and
$$\frac{1}{4} - \frac{i}{8} + \frac{3}{32}i^2 - \dots$$

will be nearly equal to the first of the neglected terms in each case, *i.e.*, to $\frac{i^2}{4}$ in the former and $-\frac{i}{8}$ in the latter series. Assuming the rate of interest to be 3 per cent., $i = {\cdot}03$, and $\frac{i^2}{4} = {\cdot}000225$; and since a_k in practice is always less than 25, the error caused by neglecting the terms $\frac{i^2}{4} - \frac{i^3}{4} + \dots$ will be less than ·0014, or can only amount to a unit in the third decimal place. Again, $\frac{i}{8} = {\cdot}00375$, so that the error caused by neglecting the terms $-\frac{i}{8} + \frac{3}{32}i^2 - \dots$ is of more importance; and if we require the third decimal place to be correct, we must take $a_k + \frac{1}{4} - \frac{i}{8}$ for the value of the annuity payable half-yearly.

Baily's formula for the value of an annuity payable quarterly is

$$a_k \times \frac{v^{\frac{3}{4}}}{16}\left\{4(1+i)^{\frac{3}{4}} + (4+i)(1+i)^{\frac{1}{2}} + 2(2+i)(1+i)^{\frac{1}{4}} + 4 + 3i\right\}$$
$$+ \frac{v^{\frac{3}{4}}}{16}\left\{3(1+i)^{\frac{1}{2}} + 2(1+i)^{\frac{1}{4}} + 1\right\},$$

which may be shown to be equal to

$$a_k\left\{1 + \frac{5i^2}{64} \dots\right\} + \frac{3}{8} - \frac{5}{32}i + \frac{15}{128}i^2 \dots$$

Here, again, the ordinary approximation, $a_k + \frac{3}{8}$, is true to the second, but not to the third decimal place.

In proving the above formulæ, Baily assumes that if i be the yearly rate of interest, then the value of a sum due in six months' time is $(1+i)^{-\frac{1}{2}}$; and of a sum due in three months, $(I+i)^{-\frac{1}{4}}$. The propriety of this assumption is called in question by Milne, and considerable difference of opinion appears to exist on the point. Without now discussing the question, it will suffice to say that my own opinion is decidedly in favour of Baily's method. Some years ago there was an animated correspondence on the subject in the *Assurance Magazine* (see Vols iii. and iv.). The argument in favour of Baily's method appears to me to have been most forcibly stated in the letter of Mr. Orchard, vol iv., p. 61, and I entirely endorse the reasoning of that letter.

The problem of finding the exact value of an annuity payable m times in each year, has been considered by Mr. Griffith Davies. He states the problem thus, p. 328 of his book:—"To find the present value of £1 annuity, payable by m equal instalments in each year, during the existence of a single life, A; supposing the interest convertible into principal at the like intervals, and the annuity to cease with the last instalment becoming due prior to the extinction of the life proposed." His solution is,

$$A^{(m)}=(P+Q)\mathbb{A}+P,$$

where $\mathbb{A}$ is the value of the £1 annuity payable yearly, calculated at that rate of interest which makes the amount of £1 in one year equal to $\left(1+\frac{i}{m}\right)^m$; also $P=\Sigma_x\frac{(m-x)m^{x-2}}{(m+i)^x}$, and $Q=\left(1+\frac{i}{m}\right)^m\Sigma_x\frac{xm^{x-2}}{(m+1)^x}$.

When we examine closely the reasoning by which this solution is obtained, we find that the suppositions made are virtually the same as those made by Baily; and accordingly the above solution must admit of being transformed into the very dissimilar looking ones given by Baily for the particular cases of half-yearly and quarterly payments.

Mr. Davies's reasoning is as follows:—The present age being k, assume, as usual, that the deaths in each year are distributed uniformly over the year, and that each year is divided into m equal portions; then the number of persons surviving the xth portion of the tth year will be

$$l_{k+t-1}-\frac{x}{m}(l_{k+t-1}-l_{k+t})=\frac{m-x}{m}l_{k+t-1}+\frac{x}{m}l_{k+t};$$

and the probability that a person of the age k will survive the xth portion of the tth year will be

$$\frac{m-x}{m}p_{k,t-1}+\frac{x}{m}p_{k,t}.$$

Also supposing interest convertible into principal m times in a year, the value of £1 due in a year's time is $\frac{1}{\left(1+\frac{i}{m}\right)^m}$, $=u$, suppose.

Then the value of the payment $\frac{1}{m}$ to be made at the end of the xth portion of the tth year will be

$$\frac{1}{m}\times\frac{u^{t-1}}{\left(1+\frac{i}{m}\right)^x}\times\left(\frac{m-x}{m}p_{k-t-1}+\frac{x}{m}p_{k,t}\right)$$

$$=\frac{(m-x)m^{x-2}}{(m+i)^x}\times u^{t-1}p_{k-t-1}+\frac{xm^{x-2}}{u(m+i)^x}\times u^t p_{k\dashv t}.$$

Therefore the value of all the payments to be made in the tth year will be

$$\Sigma_x\left\{\frac{(m-x)m^{x-2}}{(m+i)^x}\right\}u^{t-1}p_{k\dashv t-1}+\Sigma_x\left\{\frac{xm^{x-2}}{u(m+i)^x}\right\}u^t p_{k\dashv t},$$

x in the summation having the values 1, 2, 3 . . . m.

Put $$\Sigma_x\left\{\frac{(m-x)m^{x-2}}{(m+i)^x}\right\}=\mathrm{P},$$

$$\Sigma_x\left\{\frac{xm^{x-2}}{u(m+i)^x}\right\}=\mathrm{Q}.$$

Then the value of all the payments falling due in the tth year will be

$$\mathrm{P}.u^{t-1}p_{k\dashv t-1}+\mathrm{Q}u^t p_{k\dashv t}.$$

Now giving t the values 1, 2, 3 . . . , and adding, we get the total value of the annuity

$$\mathrm{P}\Sigma_t(u^{t-1}p_{k\dashv t-1})+\mathrm{Q}\Sigma^t(u^t p_{k\dashv t}).$$

But $$\Sigma_t(u^{t-1}p_{k\dashv t-1})=1+\mathbb{A},$$

and $$\Sigma_t(u^t p_{k\dashv t})=\mathbb{A},$$

where $\mathbb{A}$ is the value of an annuity of £1 payable yearly, calculated at the rate of interest which makes the value of £1 in a year equal to $\left(1+\frac{i}{m}\right)^m$. Hence, finally, the value of the annuity required is $\mathrm{A}^{(m)}=\mathrm{P}(1+\mathbb{A})+\mathrm{Q}\mathbb{A}$, or $(\mathrm{P}+\mathrm{Q})\mathbb{A}+\mathrm{P}$.

The solution is not carried beyond this point by Mr. Davies, but he deduces the values of the annuity when paid half-yearly and quarterly, viz.:—

$$\mathrm{A}^{(2)}=\left(\frac{1}{4+2i}\right)(1+\mathbb{A})+\frac{6+i}{8}\mathbb{A},$$

and $$\mathrm{A}^{(4)}=\frac{96+32i+3i^2}{4(4+i)^3}(1+\mathbb{A})+\frac{640+160i+20i^2+i^3}{1024}\mathbb{A}.$$

It is easy, however, to perform the summation indicated in the foregoing expressions for P and Q. Thus—

$$\begin{aligned}
\mathrm{P}&=\Sigma_x\frac{(m-x)m^{x-2}}{(m+i)^x}\\
&=\frac{1}{m}\,\Sigma_x\left(\frac{m}{m+i}\right)^x-\frac{1}{m^2}\,\Sigma_x x\left(\frac{m}{m+i}\right)^x\\
&=\frac{1}{m}\,\Sigma_x c^x-\frac{1}{m^2}\,\Sigma_x xc^x,\ \text{putting } c=\frac{m}{m+i}.
\end{aligned}$$

Now,

$$\begin{aligned}
\Sigma_x c^x&=c+c^2+\ .\ .\ .\ .\ +c^m\\
&=\frac{c(1-c^m)}{1-c}\\
&=\frac{m}{m+i}\cdot\frac{1-\left(\dfrac{m}{m+i}\right)^m}{1-\dfrac{m}{m+i}}\\
&=\frac{m}{i}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}=\frac{m}{i}\,(1-u);
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_x xc^x&=c+2c^2+3c^3+\ .\ .\ .\ .\ +mc^m\\
&=\frac{c(1-c^m)}{(1-c)^2}-\frac{mc^{m+1}}{1-c}\\
&=\frac{c}{(1-c)^2}\left\{1-(1+m)c^m+mc^{m+1}\right\}\\
&=\frac{\dfrac{m}{m+i}}{\left(1-\dfrac{m}{m+i}\right)^2}\left\{1-u\left(1+m-\frac{m^2}{m+i}\right)\right\}\\
&=\frac{m(m+i)}{i^2}\left\{1-u\left(1+\frac{mi}{m+i}\right)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{P}&=\frac{1}{m}\,\Sigma_x c^x-\frac{1}{m^2}\,\Sigma_x xc^x\\
&=\frac{1}{i}\,(1-u)-\frac{m+i}{mi^2}\left\{1-u\left(1+\frac{mi}{m+i}\right)\right\}\\
&=\frac{1}{i}\left(1-\frac{m+i}{mi}\right)-\frac{u}{i}\left(1-\frac{m+i}{mi}-1\right)\\
&=\frac{1}{i}\left(1-\frac{m+i}{mi}\right)+\frac{u(m+i)}{mi^2}\\
&=\frac{1}{i}-\frac{m+i}{mi^2}\,(1-u),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Q}&=\Sigma_x\frac{xm^{x-2}}{u(m+i)^x}=\frac{1}{m^2u}\,\Sigma_x\frac{xm^x}{(m+i)^x}\\
&=\frac{m+i}{mui^2}-\frac{m+i}{mi^2}\left(1+\frac{mi}{m+i}\right)\\
&=\frac{m+i}{mui^2}-\frac{1}{i}\left(\frac{m+i}{mi}+1\right);
\end{aligned}$$

and
$$\mathrm{P}+\mathrm{Q}=\frac{m+i}{mui^2}-\frac{2(m+i)}{mi^2}+\frac{u(m+i)}{mi^2}$$
$$=\frac{m+i}{mi^2u}(1-2u+u^2)$$
$$=\frac{m+i}{mi^2u}(1-u)^2.$$

So that finally we get, substituting for P and Q their values,
$$\mathrm{A}^{(m)}=\frac{m+i}{mi^2u}(1-u)^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}(1-u);$$
or, since $u=\left(\frac{m}{m+i}\right)^m$,
$$\mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}.$$

If in this equation we make $m=2$, we get for the value of the annuity payable half-yearly
$$\mathrm{A}^{(2)}=\frac{1}{i^2}\left(\frac{2+i}{2}\right)^3\left\{1-\left(\frac{2}{2+i}\right)^2\right\}^2\mathbb{A}+\frac{1}{i}-\frac{2+i}{2i^2}\left\{1-\left(\frac{2}{2+i}\right)^2\right\}$$
$$=\frac{(4i+i^2)^2}{8i^2(2+i)}\mathbb{A}+\frac{1}{i}-\frac{4i+i^2}{2i^2(2+i)}$$
$$=\frac{(4+i)^2}{8(2+i)}\mathbb{A}+\frac{1}{4+2i},$$
which is easily proved to be identical with Davies's result, as given above. The form however thus found is simpler and easier of calculation. The formula for the value of an annuity payable quarterly will be found by making $m=4$. Thus—
$$\mathrm{A}^{(4)}=\frac{1}{i^2}\left(\frac{4+i}{4}\right)^5\left\{1-\left(\frac{4}{4+i}\right)^4\right\}^2\mathbb{A}+\frac{1}{i}-\frac{4+i}{4i^2}\left\{1-\left(\frac{4}{4+i}\right)^4\right\}.$$

This is very much simpler than Davies's formula, with which it is identical, as may be shown by expanding the two, when each of them will be found equal to
$$\frac{65{,}536+49{,}152i+17{,}408i^2+3584i^3+448i^4+32i^5+i^6}{1024(4+i)^3}\times\mathbb{A}$$
$$+\frac{640+160i+20i^2+i^3}{1024}.$$

Mr. Davies finds also the value of $\mathrm{A}^{(m)}$ when m is supposed to become infinite. Resuming the equation,
$$\mathrm{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\},$$
we have
$$\left(\frac{m+i}{m}\right)^{m+1}=\frac{m+i}{m}\times\left(\frac{m+i}{m}\right)^m=\frac{m+i}{m}\left(1+\frac{i}{m}\right)^m.$$

Now supposing m to become infinite, the limit of $\frac{m+i}{m}$ or $1+\frac{i}{m}$ is 1, and the limit of $\left(1+\frac{i}{m}\right)^m$ is e^i, e being the base of the Naperian system of logarithms. Substituting then in the value of $\mathrm{A}^{(m)}$, its limit, when m becomes infinite, is

$$\frac{1}{i^2}e^i(1-e^{-i})^2\mathbb{A}+\frac{1}{i}-\frac{1}{i^2}(1-e^{-i})$$

$$=\frac{1}{i^2}(e^i-2+e^{-i})\mathbb{A}+\frac{1}{i}-\frac{1}{i^2}(1-e^{-i})$$

$$=\frac{2}{i^2}\left(\frac{i^2}{\underline{|2}}+\frac{i^4}{\underline{|4}}+\frac{i^6}{\underline{|6}}+\ldots\right)\mathbb{A}+\frac{1}{i}-\frac{1}{i^2}\left(i-\frac{i^2}{\underline{|2}}+\frac{i^3}{\underline{|3}}-\ldots\right)$$

$$=\left(1+\frac{i^2}{3.4}+\frac{i^4}{3.4.5.6}-\ldots\right)\mathbb{A}+\frac{1}{\underline{|2}}-\frac{i}{\underline{|3}}+\frac{i^2}{\underline{|4}}-\ldots$$

which is the formula given by Davies.

I now proceed to show that Davies's formulæ for the value of an annuity payable half-yearly or quarterly, are identical with Baily's. We have seen that the value of the annuity payable half-yearly is

$$\frac{(4+i)^2}{8(2+i)}\mathbb{A}+\frac{1}{4+2i}.$$

Now let j be the true annual rate of interest, then $1+j=\left(1+\frac{i}{2}\right)^2$, whence $i=2\sqrt{1+j}-2$; and substituting for i its value in terms of j, we get the value of an annuity payable half-yearly (writing a_k now for $\mathbb{A}$, since $\mathbb{A}$ is the value of the annuity calculated at the rate of interest j).

$$\frac{(2+2\sqrt{1+j})^2}{16\sqrt{1+j}}a_k+\frac{1}{4\sqrt{1+j}}$$

$$=\frac{2+j+2\sqrt{1+j}}{4\sqrt{1+j}}a_k+\frac{1}{4\sqrt{1+j}}$$

$$=\frac{\sqrt{u}}{4}(2+j+2\sqrt{1+j})a_k+\frac{\sqrt{u}}{4},$$

u being the present value of £1 due in a year at the rate of interest j. This is Baily's formula.

So again, taking the expression for the value of an annuity payable quarterly,

$$\frac{1}{i^2}\left(\frac{4+i}{4}\right)^5\left\{1-\left(\frac{4}{4+i}\right)^4\right\}^2\mathbb{A}+\frac{1}{i}-\frac{4+i}{4i^2}\left\{1-\left(\frac{4}{4+i}\right)^4\right\},$$

we shall now have $1+j=\left(1+\frac{i}{4}\right)^4$, whence $i=4(1+j)^{\frac{1}{4}}-4$, and the expression becomes, substituting as before a_k for $\mathbb{A}$,

$$\frac{(1+j)^{\frac{5}{4}}\left(1-\frac{1}{1+j}\right)^2}{16\{(1+j)^{\frac{1}{4}}-1\}^2}a_k+\frac{1}{4(1+j)^{\frac{1}{4}}-4}-\frac{4(1+j)^{\frac{1}{4}}}{64\{(1+j)^{\frac{1}{4}}-1\}^2}\left(1-\frac{1}{1+j}\right).$$

In order to transform this into the expression given by Baily, put $(1+j)^{\frac{1}{4}}=y$; then it becomes

$$\frac{y^5}{16(y-1)^2}\left(1-\frac{1}{y^4}\right)^2a_k+\frac{1}{4(y-1)}-\frac{y}{16(y-1)^2}\left(1-\frac{1}{y^4}\right)$$

$$=\frac{1}{16y^3}\left(\frac{y^4-1}{y-1}\right)^2a_k+\frac{1}{16y^3(y-1)}\left(4y^3-\frac{y^4-1}{y-1}\right)$$

$$=\frac{(y^3+y^2+y+1)^2}{16y^3}a_k+\frac{4y^3-(y^3+y^2+y+1)}{16y^3(y-1)}$$

$$=\frac{y^6+2y^5+3y^4+4y^3+3y^2+2y+1}{16y^3}a_k+\frac{3y^2+2y-1}{16y^3}$$

$$=\frac{(1+j)^{\frac{6}{4}}+2(1+j)^{\frac{5}{4}}+3(1+j)+4(1+j)^{\frac{3}{4}}+3(1+j)^{\frac{1}{2}}+2(1+j)^{\frac{1}{4}}+1}{16(1+j)^{\frac{3}{4}}}\times a_k$$

$$+\frac{3(1+j)^{\frac{1}{2}}+2(1+j)^{\frac{1}{4}}-1}{16(1+j)^{\frac{3}{4}}}$$

$$=\frac{u^{\frac{3}{4}}}{16}\{4(1+j)^{\frac{3}{4}}+(4+j)(1+j)^{\frac{1}{2}}+(4+2j)(1+j)^{\frac{1}{4}}+4+3j\}a_k$$

$$+\frac{u^{\frac{3}{4}}}{16}\{3(1+j)^{\frac{1}{2}}+2(1+j)^{\frac{1}{4}}-1\},$$

which is Baily's formula.

Resuming now the equation

$$\text{A}^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\},$$

put $1+j=\left(1+\frac{i}{m}\right)^m$; then, since $i=m(1+j)^{\frac{1}{m}}-m$, the formula becomes, writing a_k for $\mathbb{A}$, as before,

$$\frac{(1+j)^{\frac{m+1}{m}}}{m^2\{(1+j)^{\frac{1}{m}}-1\}^2}\left\{1-\frac{1}{1+j}\right\}^2a_k+\frac{1}{m\{(1+j)^{\frac{1}{m}}-1\}}$$

$$-\frac{(1+j)^{\frac{1}{m}}}{m^2\{(1+j)^{\frac{1}{m}}-1\}^2}\left\{1-\frac{1}{1+j}\right\}$$

$$=\frac{j^2a_k}{m^2(1+j)^{\frac{m-1}{m}}\{(1+j)^{\frac{1}{m}}-1\}^2}+\frac{1}{m\{1+j)^{\frac{1}{m}}-1\}}$$

$$-\frac{j}{m^2\{(1+j)^{\frac{1}{m}}-1\}^2(1+j)^{\frac{m-1}{m}}}.$$

Now

$$\frac{1}{(1+j)^{\frac{1}{m}}-1}=\frac{1}{\frac{1}{m}j-\frac{m-1}{2m^2}j^2+\frac{(m-1)(2m-1)}{1.2.3m^3}j^3-\&c.}$$

$$=\frac{m}{j}\frac{1}{1-\left(1-\frac{1}{m}\right)\frac{j}{2}+\left(1-\frac{1}{m}\right)\left(2-\frac{1}{m}\right)\frac{j^2}{6}-\&c.}$$

$$=\frac{m}{j}\left\{1+\left(1-\frac{1}{m}\right)\frac{j}{2}-\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\}.$$

Also

$$\frac{1}{\{(1+j)^{\frac{1}{m}}-1\}^2}=\frac{m^2}{j^2}\left\{1+\left(1-\frac{1}{m}\right)j+\left(1-\frac{1}{m}\right)\left(1-\frac{5}{m}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\},$$

And

$$(1+j)^{-\frac{m-1}{m}}=1-\frac{m-1}{m}j+\frac{(m-1)(2m-1)}{2m^2}j^2-\cdot\cdot\cdot$$

$$=1-\left(1-\frac{1}{m}\right)j+\left(1-\frac{1}{m}\right)\left(2-\frac{1}{m}\right)\frac{j^2}{2}\cdot\cdot\cdot$$

$$\therefore\qquad\frac{1}{(1+j)^{\frac{m-1}{m}}\{(1+j)^{\frac{1}{m}}-1\}^2}=\frac{m^2}{j^2}\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\};$$

and substituting, the value of the annuity becomes

$$\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\cdot\cdot\cdot\right\}a_k$$

$$+\frac{1}{j}\left\{1+\left(1-\frac{1}{m}\right)\frac{j}{2}-\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\}$$

$$-\frac{1}{j}\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}\cdot\cdot\cdot\right\}$$

$$=\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\cdot\cdot\cdot\right\}a_k+\frac{1}{2}\left(1-\frac{1}{m}\right)-\frac{1}{6}\left(1-\frac{1}{m^2}\right)j+\cdot\cdot\cdot$$

and neglecting terms involving j, the value of the annuity payable m times in the year is $a_k+\frac{1}{2}\left(1-\frac{1}{m}\right)$ or $a_k+\frac{m-1}{2m}$, the well known approximation.

As before, if we wish to obtain the correct value to three decimal places, we must take the next terms, the approximation then being

$$\left(1+\frac{m^2-1}{12m^2}j^2\right)a_k+\frac{m-1}{2m}-\frac{m^2-1}{6m^2}j.$$

In all that precedes it has been supposed that the deaths occurring in any year of age are distributed uniformly throughout the year. This supposition is very generally made by writers on the theory of life contingencies; but although it is very convenient and is very nearly accurate in most cases, it is well known not to be strictly correct, except where the mortality for a series of years agrees with De Moivre's hypothesis, or the numbers dying in successive years of age are equal. If on the other hand, as is generally the case, the numbers dying in successive years of age according to a table of mortality, either increase or decrease as we pass from one age to another, then the supposition fails to give satisfactory results. For if we consider the instantaneous rate of mortality during successive short intervals, we shall find that according to the hypothesis of equal distribution of the deaths, the instantaneous rate of mortality experiences sudden changes in value as we pass from one year to another; and while increasing in value throughout each year, will suddenly become less on entering the next year of age, if the total deaths in that year are less than those in the previous one. This has been pointed out (among other writers) by Mr. Gray in the *Assurance Magazine*, vol. i. pp. 142, 5, and by Professor De Morgan in the *Companion to the Almanac* for 1842. In order to obviate this *discontinuity* in the value of the instantaneous rate of mortality, it will be necessary to suppose the deaths in each year to be distributed over the year in a manner depending on the mortality of the adjacent years. Mr. Gray in the paper already referred to has supposed the deaths in each year to proceed by constant second differences depending on the mortality of the succeeding year; and Professor De Morgan has adopted the still more accurate supposition that the deaths proceed by constant third differences in a manner depending on the mortality both of the preceding and the following years. As an illustration of the extent to which the supposition of uniform distribution may err, I would quote from the Carlisle table the numbers dying at the ages 55 to 60, viz. 73, 76, 82, 93, 106, 122, and again those dying at the ages 80 to 85, viz. 116, 112, 102, 94, 84, 78.

In the next number of the *Journal*, I propose to apply the method of third differences to find more accurately the value of annuities payable half-yearly, &c.